MOTION OF ELLIPSOIDAL BUBBLE IN LOW-VISCOSITY LIQUID

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We examine the problem of bubble velocity and also the degree of bubble deformation. The bubble volume is assumed to be constant.

1. Lagrange Equations. We examine the motion of a bubble of variable form in an ideal incompressible liquid. We assume that the liquid has no free surfaces and is at rest at infinity. The bubble motion is defined by the generalized coordinates $q_1, q_2, \ldots$ and the generalized velocities $q_1^{\cdot}, q_2^{\cdot}, \ldots$.

Among the generalized coordinates q_i are the Cartesian coordinates of the bubble mass center and the parameters defining its form, the number of which is, generally speaking, infinite. It is known that the motion of a solid body in an ideal fluid is described by the Lagrange equations [1-3]. We shall use the Breakwell variational method [1] to show that the change of the generalized coordinates for a deforming bubble is also described by the Lagrange equations.

It is known [1-3] that the velocity potential Φ and the fluid velocity v depend on q_i and $q_i^{\cdot}$ and also on the space coordinates r_α, while the fluid kinetic energy T depends on q_i and $q_i^{\cdot}$ ($i = 1, 2, \ldots$). Assume that on the time interval from t_0 to t_1 the generalized coordinates change with time as $q_i(t)$ and $\delta q_i(t)$ are the variations of these coordinates, which satisfy the conditions $\delta q_i(t_0) = \delta q_i(t_1) = 0$. Then

$$\int_{t_0}^{t_1} \delta T dt = \int_{t_0}^{t_1} \left(\frac{\partial T}{\partial q_i^{\cdot}} \delta q_i^{\cdot} + \frac{\partial T}{\partial q_i} \delta q_i\right) dt = -\int_{t_0}^{t_1} \left(\frac{d}{dt}\left(\frac{\partial T}{\partial q_i^{\cdot}} - \frac{\partial T}{\partial q_i}\right) \delta q_i dt\right. \tag{1.1}$$

Here and hereafter summation over repeating subscripts is assumed. If we convert to the Lagrangian coordinates **a**, which are the coordinates of the particles of a fluid of density ρ at the time t_0, then we can obtain

$$\int_{t_0}^{t_1} \delta T\, dt = \int_{t_0}^{t_1} dt\, \delta \int \frac{\rho v^2}{2} d^3 r = \int_{t_0}^{t_1} dt\, \delta \int \frac{\rho v^2}{2} d^3 a = \int a^3 a \int_{t_0}^{t_1} \rho v_\alpha \delta v_\alpha\, dt \tag{1.2}$$

The fluid particle displacement vector $\delta \mathbf{r}$ at the time t with change of the generalized coordinates $q_i(t)$ by the magnitude $\delta q_i(t)$ is connected with the fluid particle velocity change $\delta \mathbf{v}$ by the relation

$$\left(\frac{\partial}{\partial t} \delta r_\alpha\right)_{\mathbf{a}} = \frac{d}{dt} \delta r_\alpha = \delta v_\alpha$$

If $q_i(t)$ and $\delta q_i(t)$ are given time functions, then $\delta \mathbf{r}$ is a function of **a** and t or in Eulerian coordinates a function of **r** and t.

Then from (1.2) follows

$$\int_{t_0}^{t_1} \delta T\, dt = \int d^3 a \left\{\rho v_\alpha \delta r_\alpha \Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} \rho \frac{\partial v_\alpha}{\partial t} \delta r_\alpha\, dt\right\} = \int \rho v_\alpha \delta r_\alpha \Big|_{t_0}^{t_1} d^3 r - \int_{t_0}^{t_1} dt \int \rho \frac{d v_\alpha}{dt} \delta r_\alpha d^3 r \tag{1.3}$$

It can be shown that the relation div $\delta \mathbf{r} \equiv 0$ holds for an incompressible fluid.

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 11, No. 3, pp. 76-81, May-June, 1970. Original article submitted November 3, 1969.

The equations of irrotational motion of an ideal fluid have the form

$$\rho \, d\mathbf{v} / dt = - \nabla p, \qquad \mathbf{v} = \nabla \Phi$$

The volume integrals in (1.3) reduce to surface integrals

$$\int_{t_0}^{t_1} \delta T \, dt = - \int \rho \Phi \delta r_\alpha n_\alpha \Big|_{t_0}^{t_1} dS - \int_{t_0}^{t_1} dt \int (p - p_\infty) \, \delta r_\alpha n_\alpha \, dS \tag{1.4}$$

Here p_∞ is the fluid pressure at infinity, n_α are the components of the outward normal vector to the bubble surface. It can be shown that the integrals over a sphere of infinitely large radius vanish [1].

Assume that for change of the coordinates $q_i(t)$ by $\delta q_i(t)$ the corresponding normal displacement of the bubble boundary is $w_i \delta q_i$. Then at the boundary the condition is met

$$\delta r_\alpha n_\alpha = w_i \delta q_i \tag{1.5}$$

It follows from (1.1), (1.4), (1.5) that

$$\int_{t_0}^{t_1} dt \left[\frac{d}{dt} \frac{\partial T}{\partial q_i^{\cdot}} - \frac{\partial T}{\partial q_i} - \int p w_i \, dS \right] \delta q_i = 0$$

Hence by virtue of the arbitrariness of the variations δq_i we obtain the equations

$$\frac{d}{dt} \frac{\partial T}{\partial q_i^{\cdot}} - \frac{\partial T}{\partial q_i} = \int p w_i \, dS$$

2. Equation of Motion of Bubble in Liquid of Small Viscosity. If the Reynolds number R for translational motion of the bubble exceeds one significantly, the viscous resistance forces can be included in the generalized external forces

$$\frac{d}{dt} \frac{\partial T}{\partial q_i^{\cdot}} - \frac{\partial T}{\partial q_i} = P_i \tag{2.1}$$

These forces can be found in the problem of motion of a single bubble from the rate of change of the kinetic energy of the viscous fluid. In [4] it was shown that in the case of steady bubble motion we can neglect the difference of the kinetic energy of the ideal and viscous fluids. In the following we assume that this is also valid for the motion of a bubble of varying shape.

By virtue of the squareness of T relative to the generalized velocities $q_i^{\cdot}$, it follows from (2.1) that

$$\frac{d}{dt} T = P_i q_i^{\cdot} \tag{2.2}$$

On the other hand, the kinetic energy change can be calculated from the Navier-Stokes equations

$$\frac{d}{dt} T = \int (p \delta_{\alpha\beta} - \sigma_{\alpha\beta}') \, v_\alpha n_\beta \, dS + \int U v_\alpha n_\alpha \, dS - \int \sigma_{\alpha\beta}' \frac{\partial v_\alpha}{\partial r_\beta} \, d^3 r \tag{2.3}$$

Here μ is the fluid dynamic viscosity, U is the external mass force potential. The first integral of (2.3) is the work of the external forces, which goes to increase the surface energy

$$\int (p \delta_{\alpha\beta} - \sigma_{\alpha\beta}') \, v_\alpha n_\beta \, dS = - \sigma \frac{dS}{dt} = - \sigma \frac{\partial S}{\partial q_i} q_i^{\cdot} \tag{2.4}$$

Here σ is the surface tension coefficient, S is the bubble surface area. The second integral of (2.3) is the work of the mass forces F_i per unit time

$$\int U v_\alpha n_\alpha \, dS = \int U w_i q_i^{\cdot} \, dS = F_i q_i^{\cdot} \tag{2.5}$$

The last volume integral in (2.3) is the work of the viscous forces Q_i. This integral for $R \gg 1$ can be calculated in the potential flow approximation. The admissibility of this approximation is proved in [4] for stationary bubble motion

$$- \int \sigma_{\alpha\beta}' \frac{\partial v_\alpha}{\partial r_\beta} \, d^3 r = \mu \int \frac{\partial}{\partial n} v^2 \, dS = - Q_i q_i^{\cdot} \tag{2.6}$$

Comparing (2.2) and (2.3), with the aid of (2.4)-(2.6) we find

$$P_i = -\sigma \frac{\partial S_i}{\partial q_i} + F_i - Q_i$$

If we include the surface energy σS in the Lagrange function, we obtain the equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial q_i^{\cdot}} - \frac{\partial L}{\partial q_i} = F_i - Q_i, \qquad L = T - \sigma S \tag{2.7}$$

3. Ellipsoid in Low-Viscosity Liquid. We assume that the bubble has the form of an axisymmetric ellipsoid of revolution, whose surface in the Cartesian coordinate system is described by the equation

$$(x^2 + y^2) / l_x^2 + z^2 / l_z^2 = 1 \qquad (l_z < l_x)$$

In the (α, β, φ) ellipsoidal coordinate system

$$x = k\,[(1 + \alpha^2)(1 - \beta^2)]^{1/2} \cos\varphi, \quad y = k\,[(1 + \alpha^2)(1 - \beta^2)]^{1/2} \sin\varphi,$$
$$z = k\,\alpha\beta$$

the surface of the ellipsoid corresponds to $\alpha = \alpha_0$.

Assume the ellipsoid of constant volume moves in the direction of the z axis with the velocity u and performs oscillations, retaining ellipsoidal form, so that

$$k^3 \alpha_0 (1 + \alpha_0^2) = l^3$$

Here k and α_0 are functions of the time t, l is the radius of the sphere of equivalent volume.

The normal displacements of the ellipsoid surface owing to the oscillations w_α and the translational motion w_z equal

$$w_\alpha = \frac{h_\alpha (3\beta^2 - 1)}{3(\alpha_0^2 + \beta^2)}, \quad w_z = \frac{\beta k}{h_\alpha} \qquad \left(h_\alpha = k\left(\frac{\alpha_0^2 + \beta^2}{1 + \alpha_0^2}\right)^{1/2}\right)$$

The velocity potential Φ must satisfy the Laplace equation $\Delta\Phi = 0$ with the boundary conditions

$$\frac{1}{h_\alpha}\frac{\partial \Phi}{\partial \alpha} = w_z u + w_\alpha \alpha^{\cdot} \quad \text{for} \quad \alpha = \alpha_0, \qquad \Phi \to 0 \quad \text{as} \quad \alpha \to \infty$$

Following [2], we can obtain

$$\Phi(\alpha, \beta) = \frac{uky\beta B(\alpha)}{\alpha_0 B - A} + \frac{k^2 \alpha^{\cdot}}{6}(3\beta^2 - 1)\frac{3\alpha B(\alpha) - A(\alpha)}{3yB - 1}$$
$$A(\alpha) = \operatorname{arc\,ctg} \alpha, \qquad B(\alpha) = 1 - aA(\alpha)$$
$$A = A(\alpha_0), \qquad B = B(\alpha_0), \qquad y = 1 + \alpha_0^2$$

The liquid kinetic energy T, surface area S, and energy dissipation rate E are defined by the equations

$$\frac{2}{\rho} T = -\int \frac{\Phi}{h_\alpha}\frac{\partial \Phi}{\partial \alpha}\, dS = \frac{4\pi}{3}\left(l^3 u^2 T_u + \frac{2}{15} l^5 \alpha^{\cdot 2} T_\alpha\right) \tag{3.1}$$

$$T_u = \frac{yB}{1 - yB}, \qquad T_\alpha = \frac{3\alpha B - A}{(1 - 3yB)(\alpha y)^{5/3}}$$
$$S = 2\pi l^2 s, \qquad s = \left(\frac{y}{\alpha^2}\right)^{1/3} + \frac{\ln[(1 + \sqrt{y}) / \alpha]}{(\alpha y)^{1/6}} \tag{3.2}$$

$$\frac{E}{4\pi\mu} = -\frac{1}{4\pi}\int \frac{\partial v^2}{\partial n}\, dS = -\frac{ky}{2}\frac{\partial}{\partial \alpha}\int_{-1}^{1} v^2\, d\beta = u^2 l Q_u + 4\alpha^{\cdot 2} l^3 Q_\alpha$$
$$Q_u = \frac{y^{2/3}(A + \alpha B)}{\alpha^{7/3}(A - \alpha B)^2}, \qquad Q_\alpha = \frac{q}{(1 - 3yB)^2} \tag{3.3}$$
$$q = A^2 - \frac{2A}{\alpha}\left(1 - \frac{1}{9y^2} - \frac{1}{3y} - \frac{1}{18y\alpha^2}\right) + \frac{1}{y} + \frac{1}{3y^2} + \frac{1}{9y^2\alpha^2}$$

Here and hereafter we write α in place of α_0.

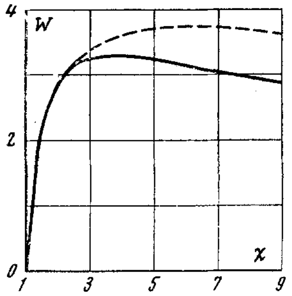

Fig. 1

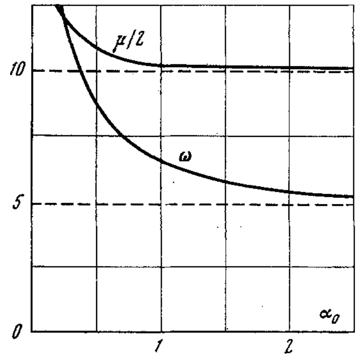

Fig. 2

For $\alpha^{\cdot} = 0$ (3.1) and (3.3) become the equations obtained in [4]. Thus, the equations of motion of the ellipsoidal bubble in a low-viscosity liquid in a gravity field with the acceleration g have the form

$$\frac{d}{dt} uT_u = g - 3\frac{\nu u}{l^2} Q_n$$

$$\frac{d}{dt}(\alpha^{\cdot} T_\alpha) - \frac{15u^2}{4l^2}\frac{dT_u}{d\alpha} - \frac{\alpha^{\cdot 2}}{2}\frac{dT_\alpha}{d\alpha} = -\frac{45}{4}\frac{\sigma}{\rho l^3}\frac{ds}{d\alpha} - 90\frac{\nu\alpha^{\cdot}}{l^2} Q_\alpha \tag{3.4}$$

4. Stationary Motion of Ellipsoidal Bubble. The steady rise of an ellipsoidal bubble is described by the equations

$$u_0 = {}^1/_3 g l^3 / \nu Q_u, \qquad T_u' = 3\sigma s' / \rho l u_0^2 \tag{4.1}$$

The second equation (4.1) defines the Weber number W as a function of the coordinate α_0 characterizing the equilibrium bubble shape

$$W = 2\rho l u_0^2 / \sigma = 6s'(\alpha_0) / T_u'(\alpha_0) \tag{4.2}$$

In [3] the dependence of the Weber number on the ellipsoid semiaxis ratio $\chi = l_x / l_z$ was obtained by satisfying the exact boundary condition $p_i - p = K\sigma$ (K is the mean surface curvature, p_i is the gas pressure in the bubble) only at the stagnation point and on the equator.

In [5] the analogous relation for $\chi \le 2$ was found numerically, with the boundary condition for the pressure being satisfied in the mean on the ellipsoid surface.

Figure 1 compares the results of [4, 5] with the function (4.2), whose curve is shown solid. Deviations from [4] show up at large χ. Thus, for $\chi \approx 5$ the difference reaches 17%. According to [4] the maximal Weber number is 3.745 at the point $\chi \approx 6$, while according to (4.2) the maximal value is 3.276 at the point $\chi \approx 3.7$.

If we exclude l and u from (4.1), we obtain the dependence of the bubble rise velocity u on l in parametric form

$$u = \left(\frac{\sigma^2 g}{12\rho^2\nu}\right)^{1/5} W^{2/5} Q_u^{-1/5}, \qquad l = \left(\frac{9\sigma\nu^2}{2\rho g^2}\right)^{1/5} W^{1/5} Q_u^{2/5} \tag{4.3}$$

Because of the weak dependence of u on W the bubble rise velocity calculated using (4.3) differs very little from Moore's results. However, in the region corresponding to $2.5 < \chi < 4$ (4.3) agrees somewhat better with experiment than do the results of [4].

5. Bubble Oscillations. Let ξ and η be the parameters characterizing the deformations and velocities of the bubble from the equilibrium position

$$\alpha = \alpha_0(1 + \xi), \quad u = u_0(1 + \eta), \quad \xi \ll 1, \quad \eta \ll 1$$

Then from (3.4) we obtain the small oscillation equations

$$T_u'\xi^{\cdot} + T_u\eta^{\cdot} = -3\varepsilon(Q_u'\xi + Q_u\eta), \qquad \varepsilon = W^{1/2}/R$$

$$T_\alpha\xi^{\cdot\cdot} - {}^{15}/_4 W(T_u''\xi + 2T_u'\eta) = -{}^{45}/_2 s''\xi - 90\varepsilon Q_\alpha\xi^{\cdot} \tag{5.1}$$

Here $R = u_0 l/\nu$ is the Reynolds number; primes denote derivatives with respect to α at the point α_0; overdots denote derivatives with respect to dimensionless time t', connected with t as follows:

$$t = t'\sqrt{2gl^3/\sigma}$$

We seek the solution of (5.1) in the form

$$\xi = \xi_0 e^{\lambda t'}, \qquad \eta = \eta_0 e^{\lambda t'}$$

Then we obtain the equation for finding λ

$$\begin{gathered}\lambda^3 T_u T_\alpha + \lambda^2 \varepsilon (3Q_u T_\alpha + 90 Q_\alpha T_u) + \lambda ({}^{15}/_2 W T_u'^2 - {}^{15}/_4 W T_u T_u'' + {}^{45}/_2 T_u s') \\ + \varepsilon ({}^{45}/_2 W T_u' Q_u' - {}^{45}/_4 W T_u'' Q_u + {}^{135}/_2 s'' Q_u) = 0\end{gathered} \tag{5.2}$$

The solution of this equation, found in the form of a series in the small parameter ε, has the form

$$\begin{gathered}\lambda = i\omega - \varepsilon\mu, \quad \omega^2 = \frac{45 s'}{2T_\alpha} F, \quad F = \frac{d}{d\alpha} \ln \frac{s' T_u^2}{T_u'} \\ \mu = \frac{3Q_u}{F T_u} \frac{d}{d\alpha} \ln \frac{T_u}{Q_u} + 45 \frac{Q_\alpha}{T_\alpha}\end{gathered} \tag{5.3}$$

Thus, we obtain the solution of the small oscillation equations

$$\begin{gathered}\alpha = \alpha_0 (1 + \xi_0 e^{-Mt} \cos \Omega t), \\ u = u_0 (1 + \eta_0 e^{-Mt} \cos \Omega t) \\ \Omega = (\sigma / 2\rho l^3)^{1/2} \omega(\alpha_0), \qquad M = \frac{\nu}{l^2} \mu(\alpha_0)\end{gathered} \tag{5.4}$$

The oscillation frequency $\omega(\alpha)$ and decay coefficient $\mu(\alpha_0)$ vary quite smoothly (Fig. 2). As $\alpha_0 \to \infty$ ($\chi \to 1$) the functions $\omega(\alpha_0)$ and $\mu(\alpha_0)$ approach asymptotically values equal to $\sqrt{24} \approx 4.90$ and 20 respectively.

Thus the ellipsoidal bubble oscillation frequency agrees in order of magnitude with the frequency of the capillary oscillations of a resting spherical bubble of equivalent volume. The oscillation amplitude decays at times of order $^1/_{20} l^2/\nu$. We note that for a highly flattened ellipsoid the liquid flow velocity increases markedly at the equator, the separation region broadens considerably, and the potential flow approximation, as shown in [4], becomes invalid.

The authors wish to thank V. G. Levich for discussions of the results obtained in this study and also D. A. Rzhanitsyn and A. S. Blokhin for assistance in the numerical calculations.

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